Math 217 Fall 2025

Quiz 8B – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Let V and W be vector spaces. A linear transformation $T : V \to W$ is said to be *invertible* if . . .

   Solution: There exists a linear transformation $S : W \to V$ such that

   $$S \circ T = \mathrm{id}_V \quad \text{and} \quad T \circ S = \mathrm{id}_W .$$

   Equivalently, T is bijective (both one-to-one and onto). The map S is necessarily unique and is called the inverse of T.

   (b) The *inverse transformation* $T^{-1} : W \to V$ is . . .

   Solution: The unique linear transformation $T^{-1} : W \to V$ satisfying

   $$T^{-1} \circ T = \mathrm{id}_V \quad \text{and} \quad T \circ T^{-1} = \mathrm{id}_W .$$

   It exists exactly when T is invertible (equivalently, bijective).

2. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

   $$T(x, y) = (2x + y,\ x - y).$$

   (a) Write T in matrix form with respect to the standard basis of $\mathbb{R}^2$.

   Solution: With respect to the standard basis, $T(\mathbf{x}) = A\mathbf{x}$ where

   $$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} .$$

*For full credit, please write out fully what you mean instead of using shorthand phrases.

(b) Determine whether T is invertible.

Solution: Compute $\det A = 2(-1) - 1 \cdot 1 = -3 \neq 0$, so A (and hence T) is invertible. An explicit inverse is

$$A^{-1} = \frac{1}{-3}\begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix},$$

so for $(u, v) \in \mathbb{R}^2$,

$$T^{-1}(u, v) = \left(\tfrac{u+v}{3}, \ \tfrac{u-2v}{3}\right).$$

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) If $T : V \to W$ and $S : W \to U$ are invertible linear transformations, then $S \circ T$ is invertible and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.

Solution: TRUE. Since T and S are bijective, so is $S \circ T$. Moreover,

$$(T^{-1} \circ S^{-1}) \circ (S \circ T) = T^{-1} \circ (S^{-1} \circ S) \circ T = T^{-1} \circ \mathrm{id}_W \circ T = \mathrm{id}_V,$$

and similarly $(S \circ T) \circ (T^{-1} \circ S^{-1}) = \mathrm{id}_U$. Hence $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.

(b) If A is an $n \times n$ matrix and B is an $n \times n$ matrix with $AB = I_n$, then $BA = I_n$ also holds.

Solution: TRUE. From $AB = I_n$, for any $\mathbf{x}$, if $B\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = A(B\mathbf{x}) = \mathbf{0}$, so B is injective and hence invertible. Let B^{-1} denote its inverse. Then $AB = I_n$ implies $A = B^{-1}$ by right-inverse uniqueness, and thus $BA = B^{-1}B = I_n$. (Equivalently, $\det(AB) = \det A \det B = 1$ forces $\det B \neq 0$, so B is invertible and $A = B^{-1}$.)